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DEVELOPING TEACHING APPROACHES TO CONCEPTS OF DIFFERENTIAL AND INTEGRAL CALCULUS WITH GEOGEBRA

Desarrollo de enfoques de enseñanza para conceptos de cálculo diferencial e integral con Geogebra

Desenvolvendo abordagens de ensino para conceitos de cálculo diferencial e integral com Geogebra

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Abstract

This paper deals with teaching approaches of function, continuity, differentiability and differential equations. It was developed four approaches for Calculus' concepts: function, differentiability, continuity and differential equation. For the concept of function, we show examples of piecewise defined functions and a function, defined by a sentence which the graphs is a curve with interruptions. The concepts of continuity and differentiability are explored by the notion of local straightness which helps to develop a formal conceptualization and it is emphasized an example of a continuous everywhere but differentiable nowhere function. The concept differential equation, $y' = f(x, y)$, is explored through a qualitative approach to seeking the solutions, which begins from the analysis of their field directions. The teaching of Calculus requires the use of materials that supports learning. The material presented in this article fulfills that requirement and it is prepared with reference to theoretical elements designed by a specialist area for this purpose. These educational approaches bring in your formatting, marks the Genesis of documents as a means of collaborating with the expansion of the set of resources for the classroom and theoretical constructs of Tall, with a view to favouring the formation of concepts mathematicians. Besides that, it is possible to identify GeoGebra's tools, commands and predefined functions that enable the development of the necessary teaching materials that are meaningful for teaching and learning concepts covered in Higher Education, especially the differential and integral calculus.

KEYWORDS: GeoGebra. Calculus and Differentiability

Resumen

Este artículo presenta los enfoques para la enseñanza de ecuaciones diferenciales, función, continuidad y differentiability. Cuatro enfoques fueron desarrollados para los siguientes conceptos del cálculo: función, continuidad y differentiability, ecuación diferencial. Para el concepto de función, se muestran ejemplos de funciones definidas por partes y una función, definida por una ecuación cuya gráfica es una curva continua no. Los conceptos de continuidad y differentiability son operados por la noción local de justicia que ayuda a desarrollar un formal y conceptualización que se acentúa en un ejemplo de una función continua y no diferenciable en todos los puntos. El concepto de ecuación diferencial, y = f (x, y), es explorado a través de un enfoque cualitativo, en la que comienza la búsqueda de soluciones de la ecuación a partir del análisis del campo de direcciones. La enseñanza de cálculo requiere el uso de materiales que apoya el aprendizaje. Los materiales presentados en este artículo cumple con este requisito, se preparan con referencia a elementos teóricos destinados a un área de

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especialista para ello. Estos enfoques educativos traen en sus formato, las marcas de la Génesis Documental como medio de colaborar con la expansión del conjunto de recursos para el aula teórica y construcciones de altura, con el fin de favorecer la formación de conceptos matemáticos. Además, es posible identificar herramientas, comandos y funciones predefinidas de GeoGebra, que permiten el desarrollo de materiales que son significativos para la enseñanza de la enseñanza y el aprendizaje de conceptos cubiertos en la educación superior, especialmente el cálculo diferencial e integral

PALABRAS CLAVE: GeoGebra. Cálculo y differentiability

Resumo
Este artigo apresenta abordagens de ensino para função, continuidade, diferenciabilidade e equações diferenciais. Quatro abordagens para conceitos de cálculo foram desenvolvidas função, diferenciabilidade continuidade e equação diferencial. Para o conceito de função, mostramos exemplos de funções definidas por várias sentenças e uma função, definida por uma sentença cuja gráfico é uma curva não contínua. Os conceitos de continuidade e diferenciabilidade são explorados pela noção de local retidão que ajuda a desenvolver uma conceituação formal e isso é enfatizado em um exemplo de uma função continua e não diferenciável em todos os pontos. O conceito de equação diferencial, y' = f (x, y), é explorado por meio de uma abordagem qualitativa, na qual a busca das soluções da equação começa a partir da análise de seu campo de direções. O ensino de Cálculo requer o uso de materiais que ofereça suporte a aprendizagem. Os materiais apresentados neste artigo cumprem esse requisito e são preparados com referência aos elementos teóricos destinados por uma área de especialista para essa finalidade. Essas abordagens de ensino trazem, em sua formatação, marcas da Gênese Documental como um meio de colaborar com a ampliação do conjunto de recursos para a sala de aula e de constructos teóricos de Tall, com vistas ao favorecimento da formação de conceitos matemáticos. Além disso, é possível identificar ferramentas, comandos e funções predefinidas do GeoGebra, que permitem o desenvolvimento de materiais didáticos que sejam significativos para o ensino e a aprendizagem de conceitos abordados no ensino superior, especialmente o cálculo diferencial e integral.

PALAVRAS-CHAVE: GeoGebra. Cálculo e diferenciabilidade

INTRODUCTION

In this article, we have shown elements of a PhD research developed in Pontifícia Universidade Católica in São Paulo. The objective of this research is to develop teaching materials for concepts of Differential and Integral Calculus, as a way to promote integration between theoretical and practical results, using the software GeoGebra.

With the objective to highlight the need to produce teaching materials for Calculus' concepts, we bring the following arguments:

Rasmussen, Marrangelle and Borba (2014) highlight that "it is fundamentally important that the body of research on calculus learning, teaching, and understanding coherently contribute to the practice of educating the millions of students who enroll in calculus courses each year" (RASMUSSEN; MARRANGELLE; BORBA, 2014, p. 507).

With regard to the researches about the learning and teaching of Calculus, Rasmussen, Marrangelle and Borba understand that their results, despite bringing contributions on the teaching and learning of the concepts limit, derivative and integral still remain isolated and uncoordinated (RASMUSSEN; MARRANGELLE; BORBA, 2014, p. 508).

Besides that Rasmussen, Marrangelle and Borba (2014) understand that due to the depth of what is known on the learning of students, obtained from earlier researches, it's necessary that researches in the Mathematical Education field, in Higher Education, engage themselves in the development of extensive researches, in which mathematics and mathematics educators work in unison to approach issues related to the learning and teaching of Calculus, of a theoretical and practical nature.

In Robert and Speer (2000) the urgency of the integration of theory and practice is reinforced in the researches of teaching Calculus. In the work of these two authors two categories of research were highlighted for the learning and teaching of Calculus. The first included researches guided by theories, and the other by researches guided by practice. This categorization didn't imply in a separation, as Robert and Speer understood these two approaches as complementary and that the research field of Mathematical Education "will make progress on effective teaching and learning only if it deals meaningfully with theoretical and pragmatic issues simultaneously" (ROBERT; SPEER, 2001, p. 297).

As so, there can be a relation established between both works analyzed in the previous paragraphs (ROBERT; SPEER,2001; RASMUSSEN; MARRANGELLE; BORBA, 2014) in which the same observation is made: the necessity of valuing the production of knowledge for the improvement of practice, in earlier researches related to the learning and teaching of Calculus.

This is one of the purposes of this research, as I intend to develop teaching materials for Calculus based on theoretical elements, proposed by David Tall and his associates, through the utilization of Documental Genesis. In my PhD. Thesis will be highlighted the construction's process of the intended material.

Theoretical framework

The theoretical framework that composes the process of material construction is: Documentational Genesis, developed by Gueudet and Trouche (2009), and the notion of Generic Organiser and Cognitive roots, developed by David Tall and his associates.

According to Gueudet e Trouche, the documentation work elaborated by teacher is the core of teachers' professional activity and professional development. This work is the following activities: looking for resources, selecting/designing mathematical tasks, planning their succession, managing available artifacts, etc.

The Documentational Genesis Process produces a document that is represented by the formula:

Document = Resources + Scheme of Utilization (1)

The word resource is utilized to describe to emphasize the variety of artifacts we consider: a textbook, a piece of software, a student's sheet, a discussion with a colleague, etc. A resource is never isolated; it belongs to a set of resources.

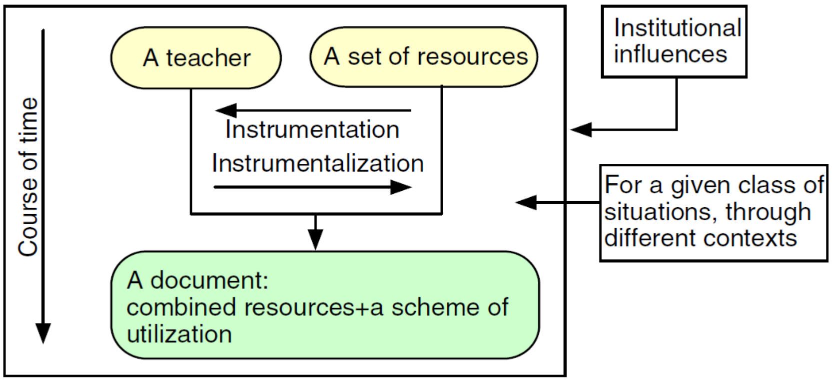
To complement this notion, Gueduet and Trouche claims that:

> [...] a resource can be an *artefact, i.e.* an outcome of human activity, elaborated for a human activity with a precise aim. But resources exceed artefacts: the reaction of a student, a wooden stick on the floor can also constitute resources, for a teacher who draws on them in her activity (GUEUDET; TROUCHE, 2012, p. 204).

The Scheme of Utilization is defined as Vergnaud (1988): it is an invariant organization subject's behavior to a class of situations (VERGNAUD, 1998).

Gueudet and Trouche represent the Documentational Genesis process for the following schematic representation:

Figure 1 – Schematic representation of a Documentational Genesis



Other theoretical elements that have referred the production of the document is the notion of generic organisers, developed by David Tall.

The notion of generic organizer is defined as "an environment (or microworld) which enables the learner to manipulate examples and (if possible) non-examples of a specific mathematical concept or a related system of concepts" (TALL, 2000, p. 10). The word "generic" means that the student's attention is directed to a particular aspect of the considered examples, and these aspects should incorporate abstract concept elements objectified by the teacher / researcher (TALL, 1986).

Tall warns that when as generic organizer is developed must be considered some elements which may be used for favoring the theoretical formal development of mathematics, as he claims

> […] a generic organiser is properly designed and the organizing agent acts effectively, the intuitive grasp of ideas offered by the organiser can provide a firm basis for the later development of the formal theory. This may depend heavily on the action of the organising agent attempting to ensure that the non-generic properties of the organiser do not act as distractors and form obstacles (TALL, 1986, p. 85).

Because of the features, available at GeoGebra, determined application built on it can be a generic organizer. However, that application must take into account the selection of an important and essential idea, which will be the focus of the student's attention. This idea is not necessarily essential to the desired mathematical theory; however, it helps the individual to develop intuitions appropriate to the theoretical development. The referenced idea is the notion of cognitive root.

As defined by Tall, a cognitive root is "as a cognitive unit which is (potentially) meaningful to the student at the time, yet contain the seeds of cognitive expansion to formal definitions and later theoretical development (TALL, 2000, p. 11).

Materials

In this work we have presented three preliminary versions of the elements that will compose the material that is intended in doctoral research project.

Functions

Tall points out that with the use of appropriate software may facilitate viewing of mathematical concepts representations with which students can develop determined significantly concept of mathematics. However, the researcher alerts for a caution that exists in the use of certain graphs plotter software which may lead the individual to develop a concept limited image seen that can be used to “draw reasonably smooth graphs given by formulae” (TALL, 1993, p. 2).

In that sense the proposed educational document presented was developed: in order to explore the idea that a function can be defined in more than a sentence or have the domain as a proper subset of the real numbers and how you can represent these functions in GeoGebra.

For instance, the function f_2: $[-2, 1] \rightarrow \mathbb{R}$, given by $f_2(x) = x^2$, it requires that the restriction in the field should be considered, you cannot directly type the function’s sentence. To plot the function f_2 the use command “*If*”.

According to the software manual (HOHENWARTER, 2009, p 41), this command has two structures: “If[<Condition>, <Then>]” and “If[<Condition>, <Then>, <Else>]”. With the boolean command “If[<Condition>, <Then>]” is possible to represent a real function graph which the range is proper subset of the real numbers. You must enter the following commands in the Graphics:

“*f*_2(*x*) = If [– 2 ≤ *x* ≤ 1, *x*^2]”

In GeoGebra, it is possible to add ≤ symbol (or ≥) by the following manner: type in the input field as follows "<=" (or "> ="); and the second manner is with the cursor in the input field, click the button located on the right side of that field, and click on the symbol ≤ (or ≥).

Figure 2 follows the graphical representation of the function f_2, in *Graphics*:

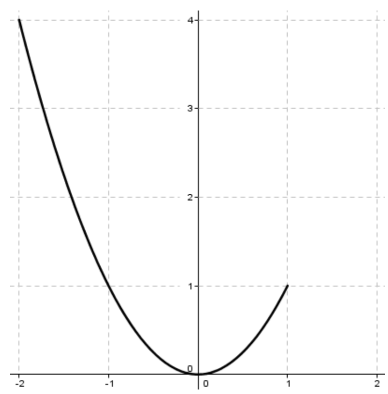


Figure 2 – A graphical representation of the function f_2: $[-2,1] \rightarrow \mathbb{R}$, given by the formulae $f_2(x) = x^2$.

Another question is found: a functional relationship must be expressed in a single mathematical sentence.

To be perceived by the student, the functional relationship can be expressed by another sentence, consider the function h: $\mathbb{R} \rightarrow \mathbb{R}$ given by the following sentence:

$$h(x) = \begin{cases} 2 - x & \text{if} \quad x \le 1 \\ x^2 & \text{if} \quad x > 1 \end{cases} \tag{2}$$

When considering the function *h*, the following question may be asked: conjecture what is the graphical representation of the function *h*?

To plot the function *h* is necessary to use other structure of the boolean command "If" given by the following structure If[<Condition>, <Then>, <Else>]". With this structure it is possible to write a real functions defined by a functional rule that has two distinct sentences as follows: all real values, which do not meet the <Condition>, satisfy the <Else> condition. So you need to enter the following in the Input:

h(*x*) = If [*x* ≤ 1, 2 – *x*, *x*^2]

The graphical representation of the function *h* is presented in Figure 3.

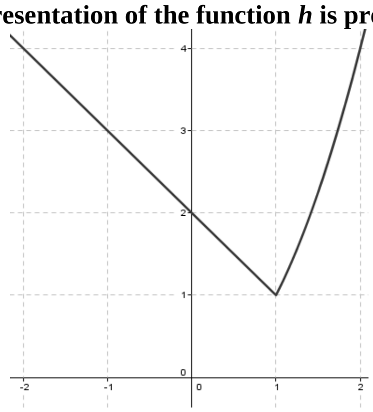


Figure 3 – The graphical representation of the function *h*: ℝ → ℝ.

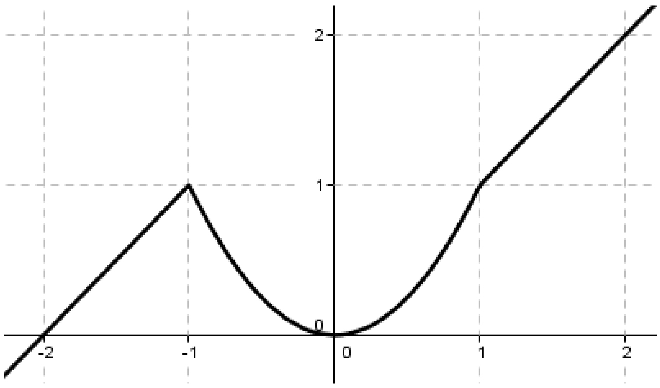
The last example of the document is a function that has three sentences. Consider the function *i*: ℝ → ℝ defined by:

$$i(x) = \begin{cases} x + 2 & \text{if} \ x < -1 \\ x^2 & \text{if} \ -1 \le x \le 1 \\ x & \text{if} \ x > 1 \end{cases} \tag{2}$$

To plot this function is necessary to "chain" two commands "If". Notice it should be entered in the Input the following commands:

"*i*(*x*) = If[*x* < –1, *x* + 2, If [–1 ≤ *x* ≤ 1, *x*^2, *x*]]"

Figure 4 – The graphical representation of the function *i*: ℝ → ℝ.



Differentiability and continuity

In Tall's formulations, the graphical representation of the differential function, when enlarged to a determined portion, looks like, locally, a segment of a straight line. Afterwards, the researcher formulated the notion of cognitive roots "local straightness", which is based on the perception that tiny part of certain graph under high magnification eventually looks virtually straight (TALL, 1989). This notion would be appropriate to the development of the concept of derivative because "it allows the gradient function to be seen as the changing gradient of the graph itself" (TALL, 1993, p. 2).

By the notion of local straightness, it would be possible to stimulate the student's imagination to conceive how a graphic representation of a continuous and non-differentiable function at the points of the domain would be. A characteristic of this representation would be the following: it should keep the "beak", not mattering how much this function is enlarged. The blancmange function would be an example of this fact, because of the way it is defined.

Concerning the concepts of continuity and differentiability of a real function, the following result is stated: Let $f\colon X \subseteq \mathbb{R} \to \mathbb{R}$ and $x_0 \in X$, if f is differentiable in x_0 then f is continuous in x_0. The reciprocity of this result is false, because there are continuous functions in a determined point of the domain, which are not differentiable at this point. In general, the counterexample to the reciprocity of the theorem is the absolute value function, that is, the real function defined by $f(x) = |x|$, in $x = 0$, it is a continuous function in 0, but not differentiable in 0, because $\lim\limits_{x\to 0} \dfrac{f(x)-f(0)}{x-0}$ does not exist. However, the differentiability is not guaranteed only at the zero point in a function. On the other hand, the blancmange function is a continuous function in all points of the domain, but not differentiable in any of those points. Besides it is difficult to conceive it and it is not commonly presented.

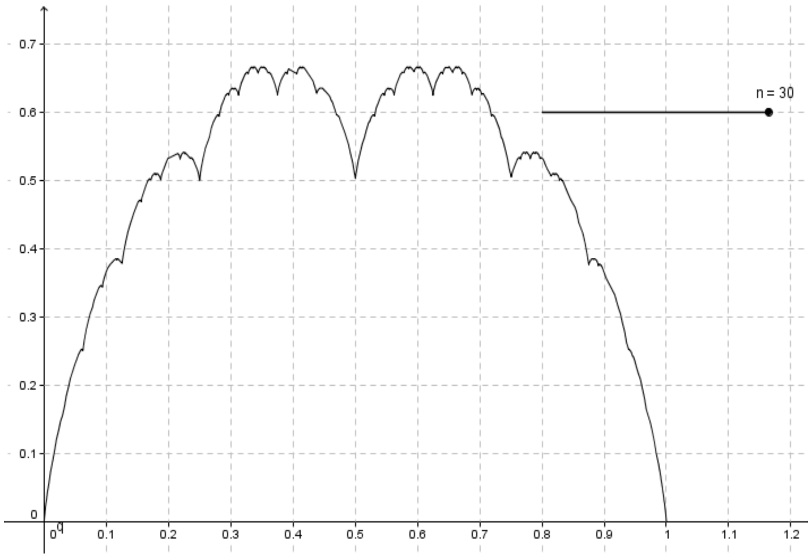
Let $f_n : [0,1] \to \mathbb{R}$, which general term of this sequence is $f_n : [0,1] \to \mathbb{R}$, with $f_n(x) = \dfrac{1}{2^{n-1}} f(2^{n-1} \cdot x)$ and $f\colon \mathbb{R} \to \mathbb{R}$, is defined by the following sentence $f(x) = d(x, \mathbb{Z})$[1]. We define the "blancmange function" as the function $b\colon [0, 1] \to \mathbb{R}$, with

[1] In the metric space $(\mathbb{R}, |\,.\,|)$, the function $d\colon \mathbb{R} \to \mathbb{R}$, $d(x, \mathbb{Z}) = inf \left\{ |x - z|, \text{ for } z \text{ an integer} \right\}$ is the function that associate each x to the distance between x and the set $\mathbb{Z}$.

$$b(x)=\sum_{i=1}^{\infty} f_i(x)$$

As a result, in the Graphics, the partial sum in the series of function is exhibited. The partial sum $\sum_{i=1}^{30} f_i(x)$, as $(f_n)_{n\in N}$ is the sequence of built functions in this session, and is represented in Figure 5.

Figure 5: the representation of the partial sum $\sum_{i=1}^{30} f_i(x)$, as $(f_n)_{n\in N}$ is the sequence of functions built in this article.



Differential Equation

The materials presented in this section have the main propose of to avoid an algebraic approach for differential equations.

Also it is remarkable the advantage of this material when we consider the Calculus' books currently edited, as Stewart (2005) and Anton, Bivens and Davis (2007), because that books use representations of direction fields to introduce differential equation. For instance, in Chapter 9 from Stewart (2005), after presenting the definition of a differential equation, the order of a differential equation and two examples of modeling, the model is a differential equation, the author presents the direction fields as a way by which to "learn a lot about the solution [of a differential equation] through a graphical approach (direction fields)" (Stewart, 2005, p. 586, adapted) even when it is not possible to obtain an analytical solution of the equation differential.

With the first material is possible to construct a direction field at a subset of $\mathbb{R}^2$ associated with a differential equation, in normal form, which is given by the following equation:

$$\frac{dy}{dx} = f(x, y) \tag{3}$$

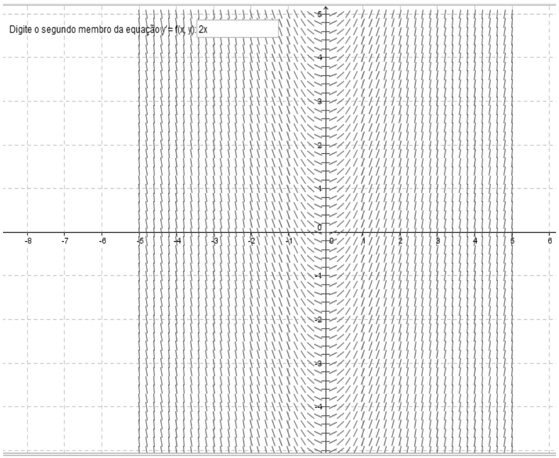
And f is a defined function in an open $A \subset \mathbb{R}^2$ assuming values in $\mathbb{R}$.

A possible interpretation of the ordinary differential equation is as follows: If (x_0, y_0) belongs to the solution curve so the tangent line to such curve at that point has slope equal to $f(x_0, y_0)$.

Thus, it is called direction field for differential equation the set of segments obtained as follows: by evaluating f at each point of a rectangular grid in $\mathbb{R}^2$. Then, at each point of the grid, a short line segment is drawn whose slope is the value of f at that point. Thus each line segment is tangent to the graph of the solution passing through that point.

For instance, consider the differential equation $y' = 2x$, we are sketching the direction field associated with it:

Figure 6 – The direction field associates to the differential equation $y' = 2x$.



Besides that, Tall suggests the presentation of this concept through the following situation:

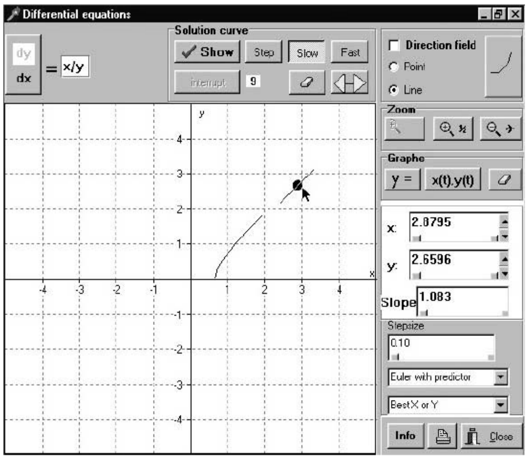
> […] consider the inverse problem to that of differentiation. (No, this is not integration!) The problem is this — if I know the gradient of a function at any point, how can I build up the graph that has that gradient? (TALL, 2000, p. 14).

And propose an embodied meaning to the concept of differential equations:

> If I point my finger at any point (x, y) in the plane, then I can calculate the gradient of the solution curve at that point as $m = f(x, y)$ and draw a short line segment of gradient m through the point (x, y) (TALL, 2000, p. 14).

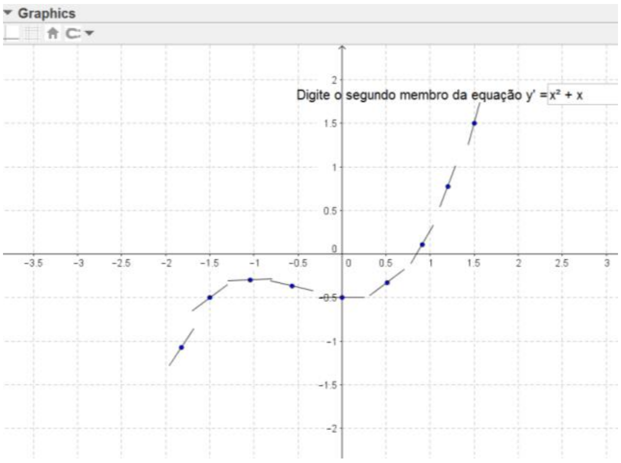
With regard to the embodied meaning proposed, the researcher developed a software that builds the graphical solution of a differential equation of 1st order as follows: the mouse is used to move a segment, whose slope is defined by the differential equation, given by the user, and with a click on the Cartesian plane, this segment is fixed (BLOKLAND, GIESSEN, TALL, 2000 *apud* TALL, 2001, p. 211).

Figure 7 – Example software which explores the solution of a differential equation.



And we have constructed a GeoGebra's tool that do same process proposed by Tall.

Figure 8 – A GeoGebra's applet inspired in Tall's application.



FINAL CONSIDERATIONS

It was presented some materials that assist the development of an educational approach based on elements developed by David Tall and collaborators to Calculus' concepts.

Another point the paper intends to show that the GeoGebra software offers tools, commands and predefined functions that enables the teacher to draw up significant educational materials for teaching and learning concepts covered in Higher Education, especially of Differential and Integral Calculus.

Furthermore, when a material is based on theories of mathematics education, your validity is justified with regard to the development of learning. In the case presented in this article, the materials presented were guided by Tall's proposals that express means of favoring learning of concepts of Calculus. But, just choose Tall's theories is not enough, because it will depend on the relevance of the choice of the elements used for the development of the material, in this case, was the software GeoGebra, that has tools and commands that allow the of the material presented.

This material brings in your formatting, marks of the Documentational Genesis as a means of collaborating with the expansion of the set of resources for the classroom and the theoretical constructs of Tall, with a view to favoring the formation of mathematical concepts. The expectation of the authors of this article in relation to the use of this material for teaching and learning is to achieve good results, on account of the theories used, however, they understand that your success will depend, as in the case of the use of any other feature, use schema.

Finally, it is expected that both the examples exposed as the tools explored can be used in developing new approaches that contribute to the advancement of Mathematics Education in Higher Education.

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